SUPERSONIC FLOW AROUND AN INCLINED CIRCULAR CONE

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The nonviscous, supersonic flow around a circular cone with a zero angle of incidence has been studied by Taylor [1] and Maccoll [2]. The flow around a cone with an angle of incidence has been examined by Stone [3-5], who sought the solution in the form of power series in α and limited himself to the terms $O(\alpha^2)$.

Ferri [6] showed that Stone's solution is not valid in the vicinity of the cone surface. He introduced the concept of a vortical layer of a thickness $O(\alpha)$, adjacent to the cone surface, whose flow parameters, except for the pressure and the normal velocity component, can differ substantially from the values given by the Stone theory. On the basis of these notions, Ferri gave correction formulas for velocity components at the cone surface with an accuracy $O(\alpha)$. Recently, Willett [7] found the velocity components at the cone surface with an accuracy $O(\alpha^2)$. He began with the assumption that the Stone theory gives the correct distribution of pressure at the cone surface and correctly determines the entropy jump during transition across a shock wave in the flow symmetry plane. Willett's assumptions, essentially, are proved by the fact that these values, calculated according to the Stone theory, correspond well to the experimental data.

Below, we will establish the above assumptions of Ferri and Willett analytically. It will be shown that, outside of the vortical layer with a thickness $O(\alpha)$, the solution is represented by an expansion of Stone's solution. Inside the vortical layer the solution is obtained with an accuracy $O(\alpha)$, which outside of this layer is transformed into the Stone solution; i.e. it will be an analytical extension of the Stone solution in the vortical layer. In this way a solution to the problem is obtained with an accuracy $O(\alpha)$ in the whole area between the cone surface and the shock wave. The behavior of the lines of constant entropy in the solution which is obtained corresponds to the qualitative analysis of Ferri [6]. It will be shown that the cone surface is special: on this surface the derivatives along the normal, of entropy S, radial u and circumferential w, comprising the velocities, become infinite (this must be taken into account in numerical methods). It will be established that the Stone theory gives w everywhere correctly (at least in the terms $O(\alpha)$).

It will be explained that, in the Stone theory, logarithmic singularities on the cone surface appear due to a break-off of series in α representing the solution, when terms $O(\alpha^2)$ are involved; when all terms with α^n (n means natural number) are considered, no such singularities are in evidence.

1. Let us examine the uniform, supersonic flow of gas around a circular one with a half-solution β at an angle of incidence in a spherical system of coordinates r, θ, φ with an axis coinciding with that of the cone (see figure).

We designate by u, v, w the velocity vector components of gas particles in a direction of growth corresponding to r, θ, φ and by p, ρ the pressure and the density. The expression for the specific entropy S has the form

$$S = c_v \ln(p / \rho^\gamma) + S_0$$

Here c_v is the specific heat capacity at a constant volume, γ the adiabatic index and S_0 the initial value of S. We examine the problem within the framework of the theory of conic flow, where u, v, w, p and ρ don not depend on r. In this case, continuity equations, quantities of movement and energies will be written in the form

$$\begin{gathered} 2\rho u \sin\theta + (\rho v \sin\theta)_\theta + (\rho w)_\varphi = 0 \\ v u_\theta + u_\varphi w \csc\theta - v^2 - w^2 = 0 \\ v v_\theta + v_\varphi w \csc\theta + \rho^{-1} p_\theta + uv - w^2 \cot\theta = 0 \\ v w_\theta + w_\varphi w \csc\theta + \rho^{-1} p_\varphi \csc\theta + w(u + v\cot\theta) = 0 \\ v (p / \rho^\gamma)_\theta + w \csc\theta (p / \rho^\gamma)_\varphi = 0 \end{gathered} \tag{1.1}$$

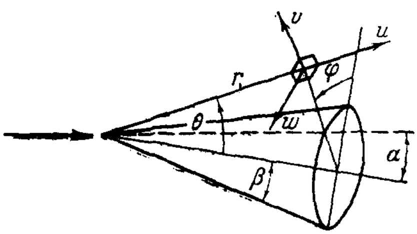
Indices θ and φ denote derivatives. Since the conic flow developed from a homogeneous current, the following Bernoulli integral is valid

$$\frac{u^2 + v^2 + w^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \frac{V_m^2}{2} \tag{1.2}$$

Here V_m is the maximum velocity of the homogeneous current. By using (1.2) we may eliminate p or ρ and obtain a system for the four functions sought. However, it is convenient to introduce, as a thermodynamic function, $s = S[\gamma(\gamma - 1)c_v]^{-1}$; we then obtain from (1.1) [6] a system of equations for u, v, w, s

$$\begin{aligned} L_1 = (a^2 - v^2)\sin\theta\, v_\theta - (w^2 - a^2)\, w_\varphi - vw(\sin\theta w_\theta + v_\varphi) - \\ - (v^2 + w^2 - 2a^2)\, u \sin\theta + a^2 v \sin\theta = 0 \\ L_2 = \sin\theta\, v u_\theta + w u_\varphi - \sin\theta\,(v^2 + w^2) = 0 \\ L_3 = \sin\theta v s_\theta + w s_\varphi = 0 \qquad \left(a^2 = \tfrac{\gamma-1}{2}(V_m{}^2 - u^2 - v^2 - w^2)\right) \\ L_4 = \sin\theta v w_\theta - a^2 s_\varphi - u u_\varphi - v v_\varphi + w(\sin\theta u + \cos\theta v) = 0 \end{aligned} \tag{1.3}$$

Here a is the immediate velocity of sound.



2. Stone [3-5] showed that the solution of the problem of a circular cone with an accuracy $O(\alpha^2)$ has the form

$$\begin{aligned} u &= u_0 + \alpha U_1 \cos\varphi + \alpha^2 (U_2 + U_3 \cos 2\varphi) \\ v &= v_0 + \alpha V_1 \cos\varphi + \alpha^2 (V_2 + V_3 \cos 2\varphi) \\ w &= \alpha W_1 \sin\varphi + \alpha^2 W_2 \sin 2\varphi \\ p / p_0 &= 1 + \alpha P_1 \cos\varphi + \alpha^2 (P_2 + P_3 \cos 2\varphi) \\ \rho / \rho_0 &= 1 + \alpha R_1 \cos\varphi + \alpha^2 (R_2 + R_3 \cos 2\varphi) \end{aligned} \tag{2.1}$$

where u_0, v_0, ..., ρ_0 are flow parameters when $\alpha = 0$; functions U_1, U_2, ..., R_3 depend only on θ. Furthermore [7]

$$s = s_0 + \alpha s_1 \cos\varphi + \alpha^2 (s_2 + s_3 \cos 2\varphi) \tag{2.2}$$

where

$$\begin{aligned} s_1 = \frac{1}{\gamma(\gamma-1)}(P_1 - \gamma R_1), \qquad s_2 = \frac{1}{\gamma(\gamma-1)}\left(\frac{\gamma}{4} R_1{}^2 - \frac{1}{4} P_1{}^2 + P_2 - \gamma R_2\right) \\ s_3 = \frac{1}{\gamma(\gamma-1)}\left(\frac{\gamma}{4} R_1{}^2 - \frac{1}{4} P_1{}^2 + P_3 - \gamma R_3\right) \end{aligned} \tag{2.3}$$

Substituting (2.1), (2.2) in $L_3 = 0$, (1.3), we obtain

$$s_{0\theta} = 0, \qquad s_{1\theta} = 0, \qquad s_{2\theta} = -s_{3\theta} = \frac{W_1 s_1}{2 v_0 \sin\theta} \tag{2.4}$$

We will henceforth designate the value which functions possess when $\theta = \beta$ by a small "x" above the line: $f(\beta) = f^{\times}$.

In the vicinity of point $\theta = \beta$ we have $v_0 = -2u_0^{\times}(\theta - \beta) + \ldots$, because, from (2.4), we obtain in this vicinity

$$s_1 = -s_3 = -\frac{W_1^{\times} s_1}{4u_0^{\times} \sin\beta} \ln(\theta - \beta) + \ldots. \tag{2.5}$$

Since the entropy in the flow must be a finite function, it is clear from (2.5) that the Stone theory is of no use in the vicinity of value $\theta = \beta$.

3. We will solve system (1.3) in the vicinity $\theta = \beta$ and require that outside of this vicinity the solution will be transformed into the Stone solution. Having found the solution, we will make the assumptions which we have justified below.

(1) The velocity components for the case of an inclined cone differ from those which obtain when $\alpha = 0$ by the amount $O(\alpha)$.

(2) The Stone theory correctly determines w with an accuracy $O(\alpha)$ everywhere. From these assumptions and from (2.1) it follows that, in the vicinity $\theta = \beta$, we may represent the component of velocity w in the form

$$w = \alpha W_1^{\times} \sin\varphi + O[\alpha(\theta - \beta)^{1/2}] + o(\alpha) \tag{3.1}$$

while from equation $L_1 = 0$ (1.3) we obtain

$$v = -2u_0^{\times}(\theta - \beta) + O[(\theta - \beta)^2] + O(\alpha)(\theta - \beta) \tag{3.2}$$

The first two terms (3.2) represent v_0; the last term appears, due to the inclination of the cone.

4. Let us find s in the vicinity $\theta = \beta$. Substituting (3.1), (3.2) in $L_3 = 0$, (1.3), we obtain the equation

$$\sin\beta\{-2u_0^{\times}(\theta - \beta) + O[(\theta - \beta)^2] + O(\alpha)(\theta - \beta)\} s_\theta + \\ + \{\alpha W_1^{\times} \sin\varphi + O[\alpha(\theta - \beta)^{1/2}] + o(\alpha)\} s_\varphi = 0$$

which, after eliminating the small values from the coefficients, we may write in the form

$$(\theta - \beta) s_\theta - \alpha h \sin\varphi\, s_\varphi = 0, \qquad h = -\frac{W_1^{\times}}{2u_0^{\times} \sin\beta} > 0 \tag{4.1}$$

An evaluation of the influence of the discarded terms will be given below. The general solution (4.1) has the form

$$s - s_0 = f(z), \qquad z = \frac{1+\cos\varphi}{1-\cos\varphi}(\theta-\beta)^{2h\alpha} \tag{4.2}$$

Since $\alpha \ln(\theta - \beta) = O(\alpha \ln \alpha)$ is an amount which is small when α is small, when $\theta - \beta = O(\alpha)$, we obtain for z an expansion to a power series by

$$z = \frac{1+\cos\varphi}{1-\cos\varphi} e^{2h\alpha \ln(\theta-\beta)} = \frac{1+\cos\varphi}{1-\cos\varphi}[1 + 2h\alpha \ln(\theta-\beta) + \ldots] \tag{4.3}$$

If the Stone theory is still valid when $\theta - \beta = O(\alpha)$, then (4.2) must transform itself into a Stone solution when $\theta - \beta = O(\alpha)$. Expanding the expressions of difference (4.2) for $s - s_0$ into a series α when $\theta - \beta = O(\alpha)$, and taking into account (4.3)

$$s - s_0 = f\left(\frac{1+\cos\varphi}{1-\cos\varphi}\right) + \ldots$$

equating the first term of this expansion to the first term of the representation for $s - s_0$ in the Stone solution (2.1), we obtain

$$f\left(\frac{1+\cos\varphi}{1-\cos\varphi}\right) = \alpha s_1 \cos\varphi, \quad \text{or} \quad f(z) = \alpha s_1 \frac{z-1}{z+1} \tag{4.4}$$

Let us now find the terms with α^2 in an expansion $s - s_0$ by powers α, when $\theta - \beta = O(\alpha)$. From (4.2) to (4.4) we obtain

$$s - s_0 = \alpha s_1 \cos\varphi + \alpha^2\left[\frac{s_1 h}{2}\ln(\theta-\beta) - \frac{s_1 h}{2}\ln(\theta-\beta)\cos 2\varphi\right] + \ldots \tag{4.5}$$

If we take into account Expression (4.1) for h, then it follows from (2.2) to (2.5) that (4.5) is identical with Expression (2.2). (Only the first terms of functions in their expansion by $\theta - \beta$ are employed in the calculation, since the following terms are determined by values which have been discarded in the derivation (4.1).)

5. Let us find u. Let us represent u in the form $u = u_0 + u_1$. From $L_4 = 0$, (1.3) and from our assumptions it follows that

$$w = \frac{a_0^{\times 2}}{u_0^{\times}\sin\beta} s_\varphi + \frac{1}{\sin\beta} u_{1\varphi} + O[\alpha(\theta-\beta)^{1/2}] + o(\alpha) \tag{5.1}$$

Substituting (5.1) in $L_2 = 0$, $L_3 = 0$, discarding small values, we obtain

$$-\sin\beta 2u_0^{\times}(\theta-\beta)u_{1\theta} + \left(\frac{a_0^{\times 2}}{u_0^{\times}\sin\beta} s_\varphi + \frac{1}{\sin\beta}u_{1\varphi}\right)u_{1\varphi} =$$
$$= \sin\beta\left(\frac{a_0^{\times 2}}{u_0^{\times}\sin\beta} s_\varphi + \frac{1}{\sin\beta}u_{1\varphi}\right)^2 \tag{5.2}$$

$$-\sin\beta 2u_0^{\times}(\theta-\beta)s_\theta + \left(\frac{a_0^{\times 2}}{u_0^{\times}\sin\beta} s_\varphi + \frac{1}{\sin\beta}u_{1\varphi}\right)s_\varphi = 0 \tag{5.3}$$

Multiplying (5.3) by $a_0^{\times 2}/u_0^{\times}\sin\beta$, and (5.2) by $\csc\beta$, and adding, we obtain

$$-\sin\beta 2u_0^{\times}(\theta-\beta)\left(\frac{a_0^{\times 2}}{u_0^{\times}\sin\beta}s_\theta+\frac{1}{\sin\beta}u_{1\theta}\right)=0$$

It follows from this that, after the expression in parenthesis has been equated to zero

$$u_1=-\frac{a_0^{\times 2}}{u_0^{\times}}(s-s_0)+\Phi(\varphi)$$

where $\Phi(\varphi)$ is the arbitrary function of φ. We will determine this function from the condition that u is transformed into a Stone solution when $\theta-\beta=O(\alpha)$. We will assume

$$\Phi(\varphi)=\alpha\left(\frac{a_0^{\times 2}}{u_0^{\times}}s_1+U_1^{+}\right)\cos\varphi$$

under which conditions

$$u_1=u-u_0=-\frac{a_0^{\times 2}}{u_0^{\times}}(s-s_0-\alpha s_1\cos\varphi)+\alpha U_1^{\times}\cos\varphi \qquad (5.4)$$

where s is given by the Formula (4.2).

Let us find the expansion u by powers α when $\theta-\beta=O(\alpha)$. Taking into account (4.5), we obtain from (5.4)

$$u=u_0+\alpha U_1^{\times}\cos\varphi+\alpha^2\left[\frac{W_1^{\times}a_0^{\times 2}s_1}{4u_0^{\times 2}\sin\beta}\ln(\theta-\beta)-\right.$$
$$\left.-\frac{W_1^{\times}a_0^{\times 2}s_1}{4u_0^{\times 2}\sin\beta}\ln(\theta-\beta)\cos 2\varphi\right]+\ldots \qquad (5.5)$$

The same results are obtained from the Stone theory. (Here also we consider the first terms in the expansion of the functions by $\theta-\beta$.) And it indeed follows from Formula (12) of [4] using the designations we are assuming here, that

$$U_2'-V_2=-U_3'+V_3=W_1\frac{U_1+W_1\sin\theta}{2v_0\sin\theta} \qquad (5.6)$$

From Formulas (19), (38) and (39) of [3] it follows that

$$U_1^{\times}+W_1^{\times}\sin\beta=-\frac{a_0^{\times 2}}{u_0^{\times}}s_1 \qquad \left(a_0^{\times 2}=\gamma\frac{p_0^{\times}}{\rho_0^{\times}},\ s_1=\frac{P_1^{\times}-\gamma R_1^{\times}}{\gamma(\gamma-1)}\right) \quad (5.7)$$

(see (2.3), (2.4)). From (5.6), (5.7), taking into account that in the vicinity $\theta=\beta$

$$V_2, V_3=O(\theta-\beta), \qquad v_0=-2u_0^{\times}(\theta-\beta)+\ldots$$

we obtain

$$U_2 = -U_3 + \cdots = \frac{W_1{}^\times a_0{}^{\times 2} s_1}{4u_0{}^{\times 2}\sin\beta}\ln(\theta-\beta) + \cdots \tag{5.8}$$

Comparing (5.5) with (2.1) and taking into account (5.8), we establish that these expressions coincide.

6. Let us now find w. It follows from (5.1), (5.4) that

$$w = -\sin\varphi\frac{\alpha}{\sin\beta}\left(\frac{a_0{}^{\times 2}}{u_0{}^\times}s_1 + U_1{}^\times\right) + O\left[\alpha(\theta-\beta)^{1/2}\right] + o(\alpha) \tag{6.1}$$

Let us find $U_1{}^\times$. If we write the Bernoulli equation (1.2) for flow around an inclined cone and subtract from it the Bernoulli equation for $\alpha = 0$, substituting into the result of the expansion (2.1) and equating to zero the coefficient for α, we obtain

$$u_0 U_1 + v_0 V_1 + \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0}(P_1 - R_1) = 0 \tag{6.2}$$

When $\theta = \beta$, it follows from (6.2) that

$$U_1{}^\times = \frac{a_0{}^{\times 2}}{\gamma-1}\frac{R_1{}^\times - P_1{}^\times}{u_0{}^\times}$$

Substituting $U_1{}^\times$, s_1 (2.3) in (6.1), we obtain

$$w = \alpha\frac{a_0{}^{\times 2}P_1{}^\times}{u_0{}^\times\gamma\sin\beta}\sin\varphi + O\left[\alpha(\theta-\beta)^{1/2}\right] + o(\alpha) \tag{6.3}$$

Let us transform w, given by the Stone theory. Equation (17) of [3], in terms of the designations we are using here, has the form

$$u_0' W_1' + (u_0 + u_0'\cot\theta)W_1 - \frac{a_0{}^2 P_1}{\gamma\sin\theta} = 0$$

From expansion $u_0{}' = v_0 = -2u_0{}^\times(\theta-\beta)$ ... in vicinity $\theta = \beta$, it follows that

$$W_1 = \frac{a_0{}^{\times 2}P_1{}^\times}{\gamma u_0{}^\times\sin\beta} + O\left[(\theta-\beta)^{1/2}\right] \qquad \text{in vicinity } \theta=\beta$$

Consequently, (2.1) may be represented in the form

$$w = \alpha\frac{a_0{}^{\times 2}P_1{}^\times}{\gamma u_0{}^\times\sin\beta}\sin\varphi + O\left[\alpha(\theta-\beta)^{1/2}\right] + o(\alpha) \tag{6.4}$$

Comparison of (6.3) and (6.4) establishes their identity.

7. Analysis of the solution we have obtained establishes that the assumptions made above are justified and that the solution, consequently, is the analytical continuation of the Stone solution in a vortical layer with a thickness $O(\alpha)$. From the results in paragraphs 4 and 5 the origin of the logarithmic singularities in terms with α^2 in expansions of u and ρ by powers α (see (2.1)) becomes clear. These singularities appear as a result of expansion into a power series in α of expressions of the type $(\theta - \beta)^{2h\alpha}$. If we consider all terms α^n (n designates a natural number) in such expansions, the logarithmic singularities disappear; i.e. the appearance of logarithmic singularities in the Stone theory is connected with the cutting-off of series expanded by powers α into terms containing α^2.

8. Let us investigate the solution we have obtained for the case of the vortical layer. From (5.4), (4.2), (4.4) it follows that at the cone surface ($\theta = \beta$)

$$u = u_0^\times + \alpha\left[\frac{a_0^{\times 2} s_1}{u_0^\times} + \left(\frac{a_0^{\times 2} s_1}{u_0^\times} + U_1^\times\right)\cos\varphi\right] + o(\alpha) \tag{8.1}$$

Substituting s_1 and U_1 in (8.1), we obtain

$$\frac{u}{u_0^\times} = 1 + \alpha\left[\frac{a_0^{\times 2}}{u_0^{\times 2}}\,\frac{P_1^\times - \gamma R_1^\times}{\gamma(\gamma-1)} - \frac{a_0^{\times 2}}{u_0^{\times 2}}\,\frac{P_1^\times}{\gamma}\cos\varphi\right] + o(\alpha) \tag{8.2}$$

When $\theta = \beta$, it follows from (6.3) that

$$\frac{w}{u_0^\times} = \alpha\,\frac{a_0^{\times 2} P_1^\times}{u_0^{\times 2}\gamma\sin\beta}\sin\varphi + o(\alpha) \tag{8.3}$$

Formulas (8.2), (8.3) coincide with Formulas (46), (52) in [7]. (They contain also terms $O(\alpha^2)$, which are also correct.) From Formulas (4.2), (4.4), (5.4), (6.3) it follows that, when $\theta = \beta$, s_θ, u_θ, w_θ become infinite.

9. Let us study the behavior of lines of constant entropy $s = \text{const}$ in the neighborhood of the cone surface. From Formulas (4.2), (4.4) we obtain

$$s - s_0 = \alpha s_1\,\frac{(1+\cos\varphi)(\theta-\beta)^{2h\alpha} + \cos\varphi - 1}{(1+\cos\varphi)(\theta-\beta)^{2h\alpha} - \cos\varphi + 1} + O(\alpha), \qquad h > 0 \tag{9.1}$$

When $\theta = \pi$ also on the cone surface, $\theta = \beta$, $s - s_0 = -\alpha s_1$; when $\varphi = 0$, we have $s - s_0 = \alpha s_1$; the other lines $s = \text{const}$ converge in a point $\theta = \beta$, $\varphi = 0$, where the Ferri peculiarity is observed, since the $s = \text{const}$ lines, according to (9.1), have the form

$$\theta - \beta = \left[\frac{1+k}{2(1-k)}\right]^{\frac{1}{2h\alpha}}\varphi^{\frac{1}{h\alpha}} + \cdots, \qquad k = \frac{s - s^\times}{\alpha s_1} = \text{const}$$

in the vicinity of point $\theta = \beta$, $\varphi = 0$.

From (5.4) we ascertain that limits u are dissimilar as they approach point $\theta = \beta$, $\varphi = 0$ along the parabolas

$$\theta - \beta = \operatorname{const} \varphi^{\frac{1}{h\alpha}} + \cdots$$

Thus, the behavior of lines of constant entropy in the above solution corresponds to that described in Ferri's [6] analysis. We should observe that the Stone theory is valid outside the layer with a thickness $O(\alpha)$ also in the vicinity of the Ferri singularity, since it follows from (9.1) that $s - s_0 = \alpha s_1 \cos\varphi + \ldots$, when $\theta - \beta = O(\alpha)$ for all cases of φ.

10. In the Stone theory, the boundary condition at the cone surface is $v = 0$ ($V_1 = V_2 = V_3 = 0$, $\theta = \beta$). In the vicinity $\theta = \beta$, the solution can not be approximated on the basis of the final segment of a power series in α. For this reason, the boundary conditions mentioned above must be substantiated. If we accept these conditions, v in the Stone theory may be represented in the form

$$v - v_0 = \alpha V_1'^{\times} (\theta - \beta) \cos\varphi + O[\alpha^2 (\theta - \beta)] + o[\alpha(\theta - \beta)] \quad \left(V_1'^{\times} = \left(\frac{dV_1}{d\theta}\right)_{\theta=\beta}\right)$$

Hence

$$v - v_0 = \alpha^2 l V_1'^{\times} \cos\varphi + o(\alpha^2), \qquad \theta - \beta = l\alpha \qquad (l = \text{const}) \qquad (10.1)$$

Let us demonstrate that, when $\theta - \beta = l\alpha$, velocity component v in the vortical layer may be determined by Formula (10.1); in the same process we will substantiate the boundary conditions $V_1 = V_2 = V_3 = 0$, when $\theta = \beta$. We will represent u, v in the vortical layer in the form $u = u_0 + u_1$, $v = v_0 + v_1$. From equation $L_1 = 0$, (1.3) it follows that

$$v_{1\theta} = \frac{w_\varphi}{\sin\beta} - 2u_1 \qquad (10.2)$$

In the right-hand part of (10.2) only those terms are left which, after integration by θ and substitution by $\theta - \beta = l\alpha$, will be $O(\alpha^2)$. Substituting in (10.2) Expressions (3.1), (5.4) for w, u_1, taking into account (4.2), (4.4), integrating the result by θ, substituting $\theta - \beta = l\alpha$ and deriving the substitute of the integration variable according to the formula $\theta - \beta = \alpha\tau$ we obtain

$$v_1 = \alpha^2 l \left(\frac{W_1^{\times}}{\sin\beta} - 2\frac{a_0^{\times 2}}{u_0^{\times}} s_1 - 2U_1^{\times}\right)\cos\varphi + $$
$$+ 2\frac{a_0^{\times 2}}{u_0^{\times}} \alpha^2 s_1 \int\limits_0^l \frac{(1+\cos\varphi)\,\alpha^{2h\alpha}\tau^{2h\alpha} - 1 + \cos\varphi}{(1+\cos\varphi)\,\alpha^{2h\alpha}\tau^{2h\alpha} + 1 - \cos\varphi}\, d\tau \qquad (10.3)$$

When $\alpha \to 0$, the integral in (10.3) tends toward $l \cos\varphi$; (10.3) can thus be written in the form

$$v - v_0 = v_1 = \alpha^2 l \left(\frac{W_1^\times}{\sin\beta} - 2U_1^\times \right) \cos\varphi + o(\alpha^2) \qquad (10.4)$$

In order for (10.4) to coincide with (10.1), we must fulfil the equality

$$V_1'^\times = \frac{W_1^\times}{\sin\beta} - 2U_1^\times$$

which, when we have accounted for the fact that $V_1 = U_1'$, must be written in the form

$$U_1'' + 2U_1 - \frac{W_1}{\sin\theta} = 0, \qquad \theta = \beta$$

Formulas (36), (39), (40), (41) of [3] indicate that this condition is in fact fulfilled, something which, indeed, did require demonstration.

11. Let us demonstrate that the Stone theory correctly determines the pressure on the cone surface with an accuracy $O(\alpha^2)$. From what we have said above, it follows that the Stone expansion represents the solution, when $\theta - \beta = O(\alpha)$. Let us estimate the change in pressure which takes place during transition through the vortical layer. The third equation (1.1) and the solution in the vortical layer indicate that p_0 may be represented in the form

$$p_\theta = -\rho v(u + v_\theta) + O(\alpha^2) = -\rho_0 v_0 (u_0 + v_0') + O[\alpha(\theta - \beta)] + O(\alpha^2) =$$
$$= p_{0\theta} + O[\alpha(\theta - \beta)] + O(\alpha^2)$$

After integrating, by θ, we discover that $p - p_0$, during its transition through the vortical layer $\theta - \beta = O(\alpha)$, varies by amounts $O(\alpha^3)$. (Willett [7] arrived at this conclusion, but his analysis has a defect, since in order to evaluate a term with v_θ, one must know the behavior of the solution in the vortical layer.) On the other hand, $p - p_0$, as given by the Stone theory, also varies by amounts $O(\alpha^3)$ during transition through the vortical layer. In actual fact

$$u_0' = v_0 = U_1' = V_1 = p_0' = \rho_0' = 0 \quad \text{when} \quad \theta = \beta$$

After differentiation by θ, (2.3), (2.4), (6.2) will now give $P_1' = R_1' = 0$, when $\theta = \beta$.

We recall that in the Stone theory P_1, P_2, P_3, R_1 are limited, while R_2, R_3 in the vicinity $\theta = \beta$ have the form

$$R_3 = -R_2 + \cdots = (\gamma - 1) \frac{s_1 W_1^\times}{4u_0^\times \sin\beta} \ln(\theta - \beta) + \cdots$$

Since $P_1' = p_0' = 0$ when $\theta = \beta$, then

$$p - p_0 = \alpha P_1 p_0 \cos\varphi + \alpha^2 (P_2 + P_3 \cos 2\varphi) p_0$$

varies by $O(\alpha^3)$ during transition through the vortical layer. The Stone theory determines p with an accuracy $O(\alpha^2)$ when $\theta - \beta = O(\alpha)$; therefore, during transition through the vortical layer, the difference between p according to the Stone theory and the exact value can be only $O(\alpha^3)$.

12. Since all of Willett's [7] assumptions have been substantiated analytically, it may be said that his formulas correctly determine velocity components at the cone surface with an accuracy $O(\alpha^2)$.

In conclusion we observe that if a solution is sought in the form of infinite series in α, these series will, apparently, converge in all cases when $\theta \neq \beta$; however, unlike series for v, w and p, series for u and ρ are divergent in actual practice, since any finite segments of these series become infinite when $\theta = \beta$.

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